Chapter 1 Introduction (Part II)

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Outline

- Set theory
- Applying set theory to probability
- Probability axioms
- Conditional probability
- Independence
- Counting methods
- Independent trials
Definition 1.7  Two Independent Events

Events A and B are independent if and only if

\[ P(AB) = P(A)P(B). \]

- When \( P[A] \neq 0 \) and \( P[B] \neq 0 \),
  - \( P[A | B] = P[A] \)
  - Learning that B occurs does not change our information about A.
  - \( P[B | A] = P[B] \)

• Review:
  \( P[A | B] = P[AB]/P[B] \)
Example 1.22  Problem
Integrated circuits undergo two tests. A mechanical test determines whether pins have the correct spacing, and an electrical test checks the relationship of outputs to inputs. We assume that electrical failures and mechanical failures occur independently. Our information about circuit production tells us that mechanical failures occur with probability 0.05 and electrical failures occur with probability 0.2. What is the probability model of an experiment that consists of testing an integrated circuit and observing the results of the mechanical and electrical tests?
Example 1.22  Solution

To build the probability model, we note that the sample space contains four outcomes:

\[ S = \{(ma, ea), (ma, er), (mr, ea), (mr, er)\} \]

where \( m \) denotes mechanical, \( e \) denotes electrical, \( a \) denotes accept, and \( r \) denotes reject. Let \( M \) and \( E \) denote the events that the mechanical and electrical tests are acceptable. Our prior information tells us that \( P[M^c] = 0.05 \) and \( P[E^c] = 0.2 \). This implies \( P[M] = 0.95 \) and \( P[E] = 0.8 \). Using the independence assumption and Definition 1.7, we obtain the probabilities of the four outcomes in the sample space as

\[
\begin{align*}
P[(ma, ea)] &= P[ME] = P[M]P[E] = 0.95 \times 0.8 = 0.76, \\
P[(ma, er)] &= P[ME^c] = P[M]P[E^c] = 0.95 \times 0.2 = 0.19, \\
P[(mr, ea)] &= P[M^cE] = P[M^c]P[E] = 0.05 \times 0.8 = 0.04, \\
P[(mr, er)] &= P[M^cE^c] = P[M^c]P[E^c] = 0.05 \times 0.2 = 0.01.
\end{align*}
\]
Definition 1.9 Events

If \( n \geq 3 \), the sets \( A_1, A_2, \ldots, A_n \) are independent if and only if

(a) every set of \( n - 1 \) sets taken from \( A_1, A_2, \ldots, A_n \) is independent,

(b) \( P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1]P[A_2] \cdots P[A_n] \).

Example 1.23 Problem

In an experiment with equiprobable outcomes, the event space is \( S = \{1, 2, 3, 4\} \). \( P[s] = 1/4 \) for all \( s \in S \). Are the events \( A_1 = \{1, 3, 4\}, A_2 = \{2, 3, 4\} \), and \( A_3 = \phi \) independent?
Example 1.23  Solution

These three sets satisfy the final condition of Definition 1.8 because $A_1 \cap A_2 \cap A_3 = \emptyset$, and

$$P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3] = 0.$$  

However, $A_1$ and $A_2$ are not independent because, with all outcomes equiprobable,

$$P[A_1 \cap A_2] = P[\{3, 4\}] = 1/2 \neq P[A_1]P[A_2] = 3/4 \times 3/4.$$  

Hence the three events are dependent.
Example 1.27  Problem

Suppose you have two coins, one biased, one fair, but you don’t know which coin is which. Coin 1 is biased. It comes up heads with probability 3/4, while coin 2 will flip heads with probability 1/2. Suppose you pick a coin at random and flip it. Let $C_i$ denote the event that coin $i$ is picked. Let $H$ and $T$ denote the possible outcomes of the flip. Given that the outcome of the flip is a head, what is $P[C_1|H]$, the probability that you picked the biased coin? Given that the outcome is a tail, what is the probability $P[C_1|T]$ that you picked the biased coin?
Example 1.27  Solution

First, we construct the sample tree.

To find the conditional probabilities, we see

$$P[C_1|H] = \frac{P[C_1 H]}{P[H]} = \frac{3/8}{3/8 + 1/4} = \frac{3}{5}.$$  

Similarly,

$$P[C_1|T] = \frac{P[C_1 T]}{P[T]} = \frac{P[C_1 T]}{P[C_1 T] + P[C_2 T]} = \frac{1/8}{1/8 + 1/4} = \frac{1}{3}.$$  

As we would expect, we are more likely to have chosen coin 1 when the first flip is heads, but we are more likely to have chosen coin 2 when the first flip is tails.
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The Number of $k$-permutations of $n$ Distinguishable Objects

Example 1.30 Problem

Shuffle the deck and choose three cards in order. How many outcomes are there?

Example 1.30 Solution

In this experiment, there are 52 possible outcomes for the first card, 51 for the second card, and 50 for the third card. The total number of outcomes is $52 \times 51 \times 50$.

Theorem 1.12

The number of $k$-permutations of $n$ distinguishable objects is

$$\begin{align*}
(n)_k &= n(n-1)(n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}.
\end{align*}$$

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Sampling without Replacement (1/2)

• We remove the object from the collection and cannot choose it again!

• \textbf{n choose k}: The number of k-combinations of n objects.

• \textbf{2 subexperiments}
  1. Choose a k-combination out of the n objects.
  2. Choose a k-permutation of the k objects in the k-combination.

• \( (n)_k = \binom{n}{k} k! \)

\textbf{Theorem 1.13}

The number of ways to choose \( k \) objects out of \( n \) distinguishable objects is

\[
\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}.
\]
Sampling without Replacement (2/2)

- We remove the object from the collection and cannot choose it again!

Example 1.32  Problem

To return to our original question of this section, suppose we draw seven cards. What is the probability of getting a hand without any queens?

Example 1.32  Solution

There are $H = \binom{52}{7}$ possible hands. All $H$ hands have probability $1/H$. There are $H_{NQ} = \binom{48}{7}$ hands that have no queens since we must choose 7 cards from a deck of 48 cards that has no queens. Since all hands are equally likely, the probability of drawing no queens is $H_{NQ}/H = 0.5504$. 
Sampling with Replacement

• Each object can be chosen repeatedly because a selected object is replaced by a duplicate!

Example 1.36

A chip fabrication facility produces microprocessors. Each microprocessor is tested to determine whether it runs reliably at an acceptable clock speed. A subexperiment to test a microprocessor has sample space $S = \{0, 1\}$ to indicate whether the test was a failure (0) or a success (1). For test $i$, we record $x_i = 0$ or $x_i = 1$ to indicate the result. In testing four microprocessors, the observation sequence $x_1x_2x_3x_4$ is one of 16 possible outcomes:

$$0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111.$$  

Theorem 1.15

For $n$ repetitions of a subexperiment with sample space $S = \{s_0, \ldots, s_{m-1}\}$, there are $m^n$ possible observation sequences.
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The probability of $n_0$ failures and $n_1$ successes in $n - n_0 + n_1$ independent trials is

$$ P[S_{n_0,n_1}] = \binom{n}{n_1} (1-p)^{n-n_1} p^{n_1} = \binom{n}{n_0} (1-p)^{n_0} p^{n-n_0}. $$

Example 1.39  **Problem**

What is the probability $P[S_{2,3}]$ of two failures and three successes in five independent trials with success probability $p$. 
Example 1.39  Solution

To find $P[S_{2,3}]$, we observe that the outcomes with three successes in five trials are 11100, 11010, 11001, 10110, 10101, 10011, 01110, 01101, 01011, and 00111. We note that the probability of each outcome is a product of five probabilities, each related to one subexperiment. In outcomes with three successes, three of the probabilities are $p$ and the other two are $1 - p$. Therefore each outcome with three successes has probability $(1 - p)^2 p^3$.

From Theorem 1.16, we know that the number of such sequences is $\binom{5}{3}$. To find $P[S_{2,3}]$, we add up the probabilities associated with the 10 outcomes with 3 successes, yielding

$$P[S_{2,3}] = \binom{5}{3}(1 - p)^2 p^3.$$
Theorem 1.19

A subexperiment has sample space \( S = \{s_0, \ldots, s_{m-1}\} \) with \( P[s_i] = p_i \). For \( n = n_0 + \cdots + n_{m-1} \) independent trials, the probability of \( n_i \) occurrences of \( s_i, i = 0, 1, \ldots, m-1 \), is

\[
P[S_{n_0,\ldots,n_{m-1}}] = \binom{n}{n_0,\ldots,n_{m-1}} p_0^{n_0} \cdots p_{m-1}^{n_{m-1}}.
\]

Definition 1.12 Multinomial Coefficient

For an integer \( n \geq 0 \), we define

\[
\binom{n}{n_0,\ldots,n_{m-1}} = \begin{cases} 
\frac{n!}{n_0!n_1!\cdots n_{m-1}!} & n_0 + \cdots + n_{m-1} = n; \\
0 & n_i \in \{0, 1, \ldots, n\}, i = 0, 1, \ldots, m-1, \\
& \text{otherwise.}
\end{cases}
\]
Example 1.42

Each call arriving at a telephone switch is independently either a voice call with probability 7/10, a fax call with probability 2/10, or a modem call with probability 1/10. Let $S_{v,f,m}$ denote the event that we observe $v$ voice calls, $f$ fax calls, and $m$ modem calls out of 100 observed calls. In this case,

$$P[S_{v,f,m}] = \binom{100}{v,f,m} \left( \frac{7}{10} \right)^v \left( \frac{2}{10} \right)^f \left( \frac{1}{10} \right)^m$$

Keep in mind that by the extended definition of the multinomial coefficient, $P[S_{v,f,m}]$ is nonzero only if $v$, $f$, and $m$ are nonnegative integers such that $v + f + m = 100$. 